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Hamiltonian theory for motions of bubbles in an infinite liquid

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The general dynamical problem for bubbles moving in an infinite expanse of perfect liquid is discussed from the standpoint of Hamiltonian theory, which is appreciated as a basis for linking symmetries with conservation laws and for identifying variational principles that describe steady motions. Allowance is made for surface tension and for an arbitrary gas law relating the pressure and volume of the bubble contents, but particular attention is paid to models where the volume is constant.

In §2, the most detailed part of the paper, a comprehensive theory is developed which represents the free surface parametrically and so applies globally in time. Conservation laws for energy and for linear and angular components of impulse are shown to follow simply from respective symmetries; consequences of Galilean invariance and of a scaling symmetry are also explored. Finally in §2, variational characterizations of steady translational, spinning and spiralling motions are explained. In §3 a formally simpler Hamiltonian theory is shown to derive from the mildly restrictive assumption that the free surface can be represented in an orthogonal coordinate system; and some special details attending the use of cylindrical coordinates are noted. For bubbles steadily translating along an axis of symmetry, approximate calculations supported by Rayleigh's principle are presented in §4.1. Steadily spiralling motions are treated in §5; estimates based on spheroidal approximations to shape are presented in §5.1; and some speculations about stability are discussed in §5.2. A brief account of generalizations dealing with multiply connected bubbles is given in §6.

1. Introduction

This paper adds to a series of investigations into Hamiltonian formulations of hydrodynamic free-boundary problems (Benjamin & Olver 1982; Olver 1983; Benjamin 1984, §6; Benjamin & Graham-Eagle 1985; Benjamin 1986*a*). The subject was initiated by Zakharov (1968) as a detail in a study of water waves, and other writers have contributed to it in the *Journal of Fluid Mechanics* (e.g. Miles 1977; Milder 1977). The particular aims here, as before, are to draw on Hamiltonian theory as, first, a frame for a systematic account of conservation laws in relation to symmetry properties and, second, a source of useful variational principles which describe steady motions and offer prospects of proving their stability (cf. Benjamin 1972, Appendix; 1974, §2; 1976). Although having been treated otherwise in the past, the present problem is ripe for illumination in these two respects.

The problem constitutes a fascinating exercise in idealized hydrodynamics. Another recommendation, long well recognized, is that ideal-fluid theory applies quite usefully to motions of bubbles subject to surface tension in liquids of small viscosity. Specifically, at Reynolds numbers greater than about 200, the Weber number comparing capillary and non-viscous hydrodynamic stresses becomes predominant as the parameter upon which stability and other observable properties depend. The incisive study by Saffman (1956) first made this point clear, and it was firmly endorsed by the experimental findings of Hartunian & Sears (1957), to which detailed reference will need to be made in §4. The relevance of ideal-fluid theory has been explored further by Moore (1965), El Sawi (1974), Miksis, Vanden-Broek & Keller (1981) and others.

Two formulations will be presented, being considered complementary rather than alternative. The first, more fully developed in §2, allows for parametric representations of the free surface. It provides a particularly neat correspondence between symmetries and the physically meaningful conservative properties of the system, also a very simple identification of the variational principles for steady motions. The second formulation, shown in §3 to derive from the first, applies only to non-parametric representations. It is superficially simpler, exemplifying Hamilton's equations in canonical form, but is less lucid in exposing all the conservation laws.

The material of §3 has affinity with a recent essay by Lewis *et al.* (1986) on the Hamiltonian structure of free-boundary problems. Their account, which extends to perfect-fluid motions with vorticity, in effect generalizes the original discovery by Zakharov (1968) and inquires deeply into geometrical aspects of the Hamiltonian formalism. Although the equations for a liquid drop *in vacuo* are taken as an example, which are of course closely related to those to be studied here, their account is considerably more abstract than the following.

2. General theory

For free motions of a gas-filled bubble in an infinite perfect liquid, a global quasi-Hamiltonian formulation of the evolutionary equations will first be noted. Hence we shall finally obtain a useful variational characterization of steady translational and spiralling motions. The account closely follows the treatment by Benjamin & Olver (1982, Appendix 1) who dealt with parametric representations of the free surface in the water-wave problem. The assumptions of the model are listed as follows:

(i) The liquid is inviscid and incompressible, with unit density, and in all directions extends to infinity where it is at rest.

(ii) The time-dependent space Δ composing the bubble is simply connected; therefore so also is the liquid-filled space $D = \mathbb{R}^3\backslash\Delta$.

(iii) The gas contained in Δ has negligible inertia; so its pressure P is uniform and is supposed to bear a known relation to the volume $\mathscr{V}$ of Δ. For present purposes there will be no need to specify a particular gas law, but the isothermal law $P\mathscr{V}$ = const. can serve as an example. An important version of the model takes $\mathscr{V}$ to be fixed. Then P, a function of time t alone, is calculable as a dynamical property.

(iv) The motion has been started from rest by the application of conservative external forces. Consequently it has a velocity potential $\phi(x_1, x_2, x_3, t)$, which because of (i) is a harmonic function of the Cartesian coordinates x_1, x_2, x_3 in D and because of (ii) is single-valued.

(v) No part of the bubble surface S, the boundary of Δ, impacts another part during the motion.

The closed surface S has infinitely many admissible parametric descriptions in the form

$$x_i = X_i(\alpha, \beta, t) \quad (i = 1, 2, 3),$$

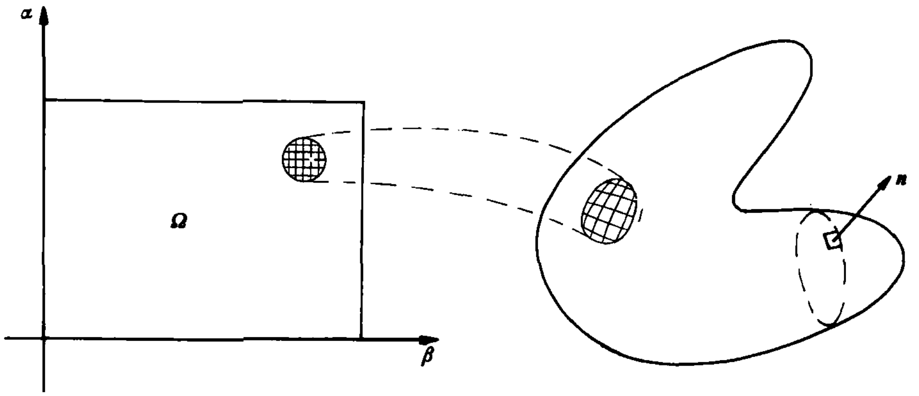

FIGURE 1. Illustration of bubble whose time-dependent surface S is described by $\mathbf{x} = \mathbf{X}(\alpha, \beta, t)$ with (α, β) ranging over fixed rectangle Ω.

and for the time being there is no need to specify any particular description. Here the X_i are taken to be periodic, twice differentiable functions of α and β, which can accordingly be supposed to range over a fixed rectangle Ω (see figure 1). In terms of

$$\gamma_1 = \frac{\partial(X_2, X_3)}{\partial(\alpha, \beta)}, \quad \gamma_2 = \frac{\partial(X_3, X_1)}{\partial(\alpha, \beta)}, \quad \gamma_3 = \frac{\partial(X_1, X_2)}{\partial(\alpha, \beta)}$$

and $$J = (\gamma_i^2)^{\frac{1}{2}} \quad \text{(summation convention)},$$

the components of the unit normal $\mathbf{n}$ to S, directed into the liquid, are γ_i/J. (Note that $J^2 = EG - F^2$ in the standard notation of differential geometry.) In general J is positive except at two points of Ω (poles on S), but the quotients γ_i/J can be assumed to have determinate limits there. (Compare the case when Δ is a torus and $J > 0$ everywhere.)

The area of S is

$$|S| = \int_S \mathrm{d}s = \int_\Omega J \,\mathrm{d}\alpha \,\mathrm{d}\beta;$$

and it is well known from differential geometry that the *first variation* of $|S|$ is given by

$$|\dot{S}| \equiv \left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_\Omega J(X + \epsilon\dot{X}) \,\mathrm{d}\alpha \,\mathrm{d}\beta\right]_{\epsilon=0} = \int_\Omega 2H\gamma_i \dot{X}_i \,\mathrm{d}\alpha \,\mathrm{d}\beta,$$

where H is the mean curvature of S (positive where S is curved inwards). Thus, defining variational derivatives in respect of the inner product for $L^2(\Omega)$, we have

$$\frac{\delta|S|}{\delta X_i} = 2H\gamma_i. \tag{2.1}$$

The following notation will be helpful:

$$\Phi = \phi_S = \phi(X_1, X_2, X_3, t),$$

$$\Phi_{(t)} = \left(\frac{\partial\phi}{\partial t}\right)_S, \quad \Phi_{(i)} = \left(\frac{\partial\phi}{\partial x_i}\right)_S,$$

$$\Phi_{(n)} = \left(\frac{\partial\phi}{\partial n}\right)_S = J^{-1}\gamma_i \Phi_{(i)}, \quad q^2 = \Phi_{(i)}^2.$$

Note that

$$\frac{\partial \Phi}{\partial t} = \Phi_{(t)} + \Phi_{(i)} \frac{\partial X_i}{\partial t}, \tag{2.2}$$

and

$$\dot{\Phi} = (\dot{\phi})_s + \Phi_{(i)} \dot{X}_t. \tag{2.3}$$

The second identity shows how the first (infinitesimal) variation $\dot{\Phi}$ of Φ is derived from that of ϕ in D and that of $\boldsymbol{X}$.

2.1. *The hydrodynamic problem*

The velocity potential ϕ is required to be a harmonic function $D(t) \to \mathbb{R}$ vanishing together with $|\nabla \phi|$ as $|\boldsymbol{x}| \to \infty$. At each t, given $D(t)$ and Φ, therefore ϕ is uniquely determined as the solution of a linear Dirichlet problem; and so the normal velocity $\Phi_{(n)}$ of the free surface S is implied as a functional transformation of Φ.

The kinematic boundary condition at S is

$$\gamma_i \frac{\partial X_i}{\partial t} = J\Phi_{(n)} = \gamma_i \Phi_{(i)}, \tag{2.4}$$

ensuring that the normal velocity of the moving surface equals the normal velocity of the liquid. The dynamical boundary condition at S is expressible by the Bernoulli integral of the Euler equations for irrotational motion, thus

$$\Phi_{(t)} + \tfrac{1}{2}q^2 - 2\sigma H + P = 0, \tag{2.5}$$

where σ is the coefficient of surface tension and P is the uniform pressure inside the bubble relative to the pressure in the liquid at infinity. Note that in (2.5) $P - 2\sigma H$ is the pressure in the liquid at the bubble surface.

To identify the generalized Hamiltonian structure of the system constituted by (2.4), (2.5) and the associated potential problem, we can proceed without particularizing the choice of parametric representation for S. One possible choice deserving mention, however, is to define α and β as Lagrangian coordinates that specify fluid particles lying in S. In this case we have $\partial X_i / \partial t = \Phi_{(i)}$ for each i. But for present purposes a Lagrangian description of S has little advantage.

2.2. *The energy functional*

The potential energy of the system is $\sigma|S| - \int P \, \mathrm{d}\mathscr{V}$, where P is known as a function of $\mathscr{V}$ alone from the prescribed gas law. The lower limit of the integral is of course arbitrary and immaterial. The kinetic energy K is expressed by

$$2K = \int_D |\nabla \phi|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = -\int_\Omega \Phi_{(n)} \Phi J \, \mathrm{d}\alpha \, \mathrm{d}\beta,$$

in which the second expression follows from the first according to Green's theorem and $\mathrm{d}s = J \, \mathrm{d}\alpha \, \mathrm{d}\beta$.

The first variation of K is seen to be

$$\begin{aligned} \dot{K} &= \int_S (-\tfrac{1}{2}q^2) \, n_i \dot{X}_i \, \mathrm{d}s + \int_D \nabla \phi \cdot \nabla \dot{\phi} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \\ &= \int_\Omega (-\tfrac{1}{2}q^2) \, \gamma_i \dot{X}_i \, \mathrm{d}\alpha \, \mathrm{d}\beta - \int_\Omega \Phi_{(n)} (\dot{\phi})_S J \, \mathrm{d}\alpha \, \mathrm{d}\beta. \end{aligned}$$

(Note that the reduction from the first to the second line by means of Green's theorem depends on the assumption that the variation ϕ of ϕ is small enough at large $r = |x|$ for there to be no contribution from the implicit surface integral at infinity. Contrary cases will need to be considered presently, however, in connection with the infinitesimal generators of certain symmetry groups.) Hence, using (2.1) and (2.3), we conclude that the variational derivatives of the total energy $E = K+\sigma|S|-\int P\,\mathrm{d}\mathscr{V}$ are

$$\left.\begin{aligned}\frac{\delta E}{\delta X_i} &= (-\tfrac{1}{2}q^2+2\sigma H-P)\,\gamma_i+J\Phi_{(n)}\,\Phi_{(i)},\\ \frac{\delta E}{\delta \Phi} &= -J\Phi_{(n)},\end{aligned}\right\} \tag{2.6}$$

which will be considered collectively as a four-component column vector written $\operatorname{grad} E(X_1, X_2, X_3, \Phi)$.

2.3. *Hamiltonian formulation*

The solution of the hydrodynamic problem can be represented as the vector-valued variable

$$\boldsymbol{U} = [X_1, X_2, X_3, \Phi]^{\mathrm{T}} = \boldsymbol{U}(\alpha, \beta, t).$$

This representation is plainly legitimate since the X_i and $\Phi = \phi_S$ fully determine the motion at each instant. Hence the hydrodynamic problem is seen to be expressed by

$$\boldsymbol{M}\frac{\partial \boldsymbol{U}}{\partial t} = \operatorname{grad} E(\boldsymbol{U}), \tag{2.7}$$

where $\boldsymbol{M}$ is the skew-symmetric matrix defined as follows. In terms of

$$a_{ij} = \gamma_i \Phi_{(j)} - \gamma_j \Phi_{(i)} = -a_{ji},$$

the definition is

$$\boldsymbol{M} = \begin{bmatrix} 0 & a_{21} & a_{31} & \gamma_1 \\ a_{12} & 0 & a_{32} & \gamma_2 \\ a_{13} & a_{23} & 0 & \gamma_3 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & 0 \end{bmatrix}. \tag{2.8}$$

Confirmation of the equivalence between (2.7) and the nonlinear boundary conditions is straightforward (cf. Benjamin & Olver 1982, pp. 178–9). In view of (2.6), the kinematic condition (2.4) is recovered by the fourth row of (2.7), and the first three rows are (with $i = 1, 2, 3$)

$$-a_{ij}\frac{\partial X_j}{\partial t}+\gamma_i\frac{\partial \Phi}{\partial t} = (-\tfrac{1}{2}q^2+2\sigma H-P)\,\gamma_i+J\Phi_{(n)}\,\Phi_{(i)}.$$

Multiplying by γ_i/J^2, summing over i and then substituting for $J\Phi_{(n)}$ from (2.4) as already inferred from (2.7), we obtain

$$\frac{\partial \Phi}{\partial t}-\Phi_{(i)}\frac{\partial X_i}{\partial t}+\tfrac{1}{2}q^2-2\sigma H+P = 0,$$

which because of (2.2) is the same as (2.5). Conversely, given (2.4) and (2.5), substitution for $\partial X_i/\partial t$ and for $\partial \Phi/\partial t$ as expressed by (2.2) verifies each row of (2.7) in the light of (2.6).

It should be emphasized that the matrix $\boldsymbol{M}$ depends on U through the coefficients γ_i and a_{ij}, also that det $(\boldsymbol{M}) = 0$ so that $\boldsymbol{M}$ is not invertible in any elementary sense.† Thus (2.7) is only 'quasi-Hamiltonian' and in general irreducible globally in time to canonical Hamiltonian form, although the latter will be exemplified by the simpler but restricted theory to be presented in §3. There will be no need here to go into full mathematical details of the Hamiltonian structure. Equation (2.7) as it stands is nevertheless well suited to present aims. First, it frames the connection between symmetries of the physical system and otherwise easily confirmable conservative properties. Second and more important, it reveals the variational principles characterizing steady motions.

The first aim is met by adapting a generalization of Noether's theorem that has been developed by Olver (1980, §5; 1983; cf. also Benjamin & Olver 1982, §5). The needed rule in abstract is stated as follows without proof, but it will be borne out in interesting fashion by the conservation laws to be discussed below.

Let the four-component vector $\boldsymbol{V}$ represent the infinitesimal generator of a one-parameter symmetry group for (2.7), in the sense that if $\boldsymbol{U}(\alpha, \beta, t)$ is any solution $\boldsymbol{U} + \epsilon \boldsymbol{V}$ is also a solution to $O(\epsilon)$. Let the Hamiltonian E be unaffected to $O(\epsilon)$ by the transformation represented by $\epsilon \boldsymbol{V}$. Then a functional f found to satisfy

$$\boldsymbol{M}\boldsymbol{V} = \operatorname{grad} f(\boldsymbol{U}) \tag{2.9}$$

is independent of t when evaluated on any solution of (2.7).

A different conclusion holds in the case that a modified Hamiltonian, say $\mathscr{H}$, depends explicitly on t or $\boldsymbol{x}$ and is thereby affected to $O(\epsilon)$ by the transformation. Then the property implied by (2.9) is that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left[\frac{\partial \mathscr{H}}{\partial \epsilon}\right]_{\epsilon=0} \tag{2.10}$$

for any solution of (2.7).

A more subtle elaboration is needed for Galilean transformations and one other, which are such that the first-order perturbation $\epsilon\hat{\phi}$ of ϕ does not vanish in the limit $|\boldsymbol{x}| \to \infty$. The implication of (2.9) in such a case can be proved in abstract without too much difficulty, although with a necessary generalization of the meaning attached to grad K. Being more readily verifiable by a direct calculation for the specific symmetries in question, the implication is that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{r\to\infty} \int_{\mathscr{S}_r} \left(\hat{\phi}\frac{\partial \phi}{\partial r} - \phi\frac{\partial \hat{\phi}}{\partial r}\right) \mathrm{d}s. \tag{2.11}$$

Here $\mathscr{S}_r$ denotes the sphere $x_1^2 + x_2^2 + x_3^2 = r^2$.

2.4. *Invariants*

Energy

Since neither E nor $\boldsymbol{M}$ depends explicitly on t, the most obvious symmetry group for (2.7) is that of translation along the t-axis, its infinitesimal representative being

† But at a deeper level it can be appreciated that $\boldsymbol{M}$ is invertible after the function class for U has been factorized by the group of all parametric representations. I am indebted to Professor J. E. Marsden for this observation.

$V = -\partial U/\partial t$. So E is invariant according to the rule (2.9). The property of energy conservation is verified directly by consideration that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_\Omega (\operatorname{grad} E)\cdot \boldsymbol{U}_t \,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$= \int_\Omega (\boldsymbol{M}\boldsymbol{U}_t)\cdot \boldsymbol{U}_t \,\mathrm{d}\alpha\,\mathrm{d}\beta = 0,$$

because $\boldsymbol{M}$ is skew-symmetric.

Impulse

Invariants associable with spatial symmetries of the system, namely translations and rotations, can be anticipated to constitute components of Kelvin impulse (Lamb 1932, §§120 and 121; Kochin, Kibel & Roze 1964, p. 397; Birkhoff 1950, chapter 5). But it will be of interest to see how they conform with the rule (2.9). The linear impulse of the motion is the vector

$$\boldsymbol{I} = -\int_S \Phi \boldsymbol{n}\,\mathrm{d}s, \tag{2.12a}$$

whose components are

$$I_i = -\int_\Omega \Phi\gamma_i \,\mathrm{d}\alpha\,\mathrm{d}\beta; \tag{2.12b}$$

and the corresponding impulsive couple is

$$\boldsymbol{L} = -\int_S \Phi(\boldsymbol{X}\times\boldsymbol{n})\,\mathrm{d}s, \tag{2.13a}$$

whose components are

$$L_i = -\int_\Omega \Phi\epsilon^{ijk}X_j\gamma_k \,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{2.13b}$$

It is not difficult to verify directly from (2.4) and (2.5) that $\boldsymbol{I}$ and $\boldsymbol{L}$ are constant vectors. Alternatively, with reference to the rule (2.9), we note for example that

$$\operatorname{grad} I_1 = [0, \partial(\Phi, X_3), \partial(X_2, \Phi), -\gamma_1]^{\mathrm{T}},$$

where $\partial(.,.)$ is short for $\partial(.,.)/\partial(\alpha,\beta)$ and where the definition $\gamma_1 = \partial(X_2, X_3)$ and integrations by parts have been used to reduce I_1. Since $\Phi = \phi(X_1, X_2, X_3, t)$ it follows that

$$\partial(\Phi, X_3) = \Phi_{(1)}\partial(X_1, X_3) + \Phi_{(2)}\partial(X_2, X_3)$$
$$= -\Phi_{(1)}\gamma_2 + \Phi_{(2)}\gamma_1 = a_{12},$$

and similarly $\qquad \partial(X_2, \Phi) = a_{13}.$

Thus
$$\operatorname{grad} I_1 = [0, a_{12}, a_{13}, -\gamma_1]^{\mathrm{T}}$$
$$= \boldsymbol{M}[1,0,0,0]^{\mathrm{T}}$$

in the light of (2.8). But $V = [1,0,0,0]^{\mathrm{T}}$ plainly generates the symmetry group of translations in the x_1-direction: the addition of any number ϵ to X_1 merely shifts the bubble by this distance in the x_1-direction and leaves the dynamical problem unchanged. In particular, E is unaffected by the transformation. According to (2.9), therefore, I_1 is a constant of the motion. The invariance of I_2 and I_3 is implied similarly.

As regards $\boldsymbol{L}$, consider for example

$$L_1 = -\int_\Omega \Phi(X_2\gamma_3 - X_3\gamma_2)\,\mathrm{d}\alpha\,\mathrm{d}\beta,$$

which is found to give

$$\begin{aligned}\operatorname{grad} L_1 &= [a_{12}X_3 - a_{13}X_2,\ -a_{23}X_2,\ -a_{23}X_3,\ \gamma_2X_3-\gamma_3X_2]^{\mathrm{T}}\\ &= \boldsymbol{M}[0,-X_3,X_2,0]^{\mathrm{T}}.\end{aligned} \tag{2.14}$$

Here $\boldsymbol{V} = [0, -X_3, X_2, 0]^{\mathrm{T}}$ represents the infinitesimal generator of the rotation group about the x_1-axis. So the rule (2.9) confirms L_1 to be a constant of the motion. Similar conclusions apply to L_2 and L_3.

Volume changes

Another obvious symmetry of (2.7) is that any number ϵ can be added to ϕ and Φ without changing the problem. In this case we have $\boldsymbol{V} = [0,0,0,1]^{\mathrm{T}}$ and so

$$\boldsymbol{M}\boldsymbol{V} = [\gamma_1,\gamma_2,\gamma_3,0]^{\mathrm{T}} = \operatorname{grad}\mathscr{V}.$$

Since $\hat{\phi} = 1$, the formula (2.11) consequent upon (2.9) reproduces the simple kinematic identity

$$\frac{\mathrm{d}\mathscr{V}}{\mathrm{d}t} = \int_\infty \frac{\partial\phi}{\partial r}\,\mathrm{d}s = -4\pi A_0,$$

where A_0 is the monopole coefficient of the far field (see (2.17) below).

Dynamic-kinematic combinations

The next property deserving attention relates to the Galilean invariance of the system (2.7). Let us first note the infinitesimal generator of the pertinent symmetry group. In respect of the x_1-direction it is represented by

$$\boldsymbol{V} = [t,0,0,X_1]^{\mathrm{T}}; \tag{2.15}$$

for if $\boldsymbol{U}$ is any solution with corresponding velocity potential $\phi(x_1,x_2,x_3,t)$, the infinitesimal perturbation

$$\boldsymbol{U} + \epsilon[t,0,0,X_1]^{\mathrm{T}}, \quad \phi(x_1-\epsilon t, x_2, x_3, t) + \epsilon x_1$$

is evidently also a solution to $O(\epsilon)$. The new solution just corresponds to the original motion observed in a frame of reference moving at constant velocity $-\epsilon$ in the x_1 direction. We find for the case (2.15) that

$$\begin{aligned}\boldsymbol{M}\boldsymbol{V} &= t\operatorname{grad} I_1 + [\gamma_1X_1,\gamma_2X_1,\gamma_3X_1,0]^{\mathrm{T}}\\ &= t\operatorname{grad} I_1 + \operatorname{grad} C_1,\end{aligned}$$

where

$$C_1 = \int_\Delta x_1\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \bar{x}_1\mathscr{V}. \tag{2.16}$$

Thus (2.9) is exemplified with $f = tI_1 + C_1$.

In the present case the result (2.11) is again operative, and we have $\hat{\phi} = x_1$ plus terms vanishing as $r\to\infty$. In terms of spherical polar coordinates (r,θ,ψ), the asymptotic form of the original velocity potential is

$$\phi = \frac{A_0}{r} + \frac{1}{r^2}(A_1\cos\theta + A_2\sin\theta\cos\psi + A_3\sin\theta\sin\psi) + O\left(\frac{1}{r^3}\right), \tag{2.17}$$

where the monopole coefficient A_0 and dipole coefficients A_i are functions of t alone. The coordinates being chosen so that $x_1 = r\cos\theta$, only the dipole term with coefficient A_1 makes a non-zero contribution to the integral on the right side of (2.11). Since $I_1 = \text{const.}$ it thus appears that

$$I_1 + \frac{\mathrm{d}C_1}{\mathrm{d}t} = -4\pi A_1. \tag{2.18}$$

The same argument leads to corresponding results with subscripts 2 and 3 in place of 1.

Although its connection with the Galilean symmetry group is of particular interest from the present standpoint, the result (2.18) is otherwise immediately deducible from Green's theorem, the definition of I_1 and the obvious identity

$$\frac{\mathrm{d}C_1}{\mathrm{d}t} = \int_S \Phi_{(n)}\, X_1\,\mathrm{d}s.$$

Note that (2.18) is independent of precise conditions at infinity, as are needed to make the total momentum of the fluid determinate and to account fully for the relation between gross dynamic properties and gross kinematic ones such as drift (cf. Benjamin 1986*b*).

For water waves in the case of infinite depth, the result corresponding to (2.18) was shown in Benjamin & Olver (1982, §6.5) to have a simple consequence because generally $A_1 = 0$. But the present problem admits no such simplification, and (2.18) is not comparably informative about the kinematics of bubble motions. It nevertheless provides a mildly noteworthy sidelight on a proposition due to Saffman (1967), who showed that a deformable massless body can propel itself over any distance in an infinite perfect fluid. Over such a motion starting and ending at rest, the integral of I must be zero, since no external force acts. So (2.17) indicates that the net change in C is wholly accountable to the deformations producing a non-zero average value of the dipole far field.

Virial

Another property worth attention concerns the scalar

$$W = -\int_S \Phi(\boldsymbol{X}\cdot\boldsymbol{n})\,\mathrm{d}s = -\int_\Omega \Phi\gamma_i X_i\,\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{2.19}$$

which may be called the virial of the motion (cf. Benjamin & Olver 1982, §§6.2–6.4). The property relates to the scaling symmetry for (2.7), as represented by the fact that in the case $\mathrm{d}\mathscr{V}/\mathrm{d}t = 0$ (see below) any given solution transforms into new solutions

$$\boldsymbol{X}^* = \lambda\boldsymbol{X}(\alpha,\beta,\lambda^{-\frac{3}{2}}t), \quad \Phi^* = \lambda^{\frac{1}{2}}\Phi(\alpha,\beta,\lambda^{-\frac{3}{2}}t),$$

$$\phi^* = \lambda^{\frac{1}{2}}\phi(\lambda^{-1}\boldsymbol{x},\lambda^{-\frac{3}{2}}t), \quad P^* = \frac{P}{\lambda},$$

where λ is any positive number. Details of the relation are complicated, however, and little is missed by ignoring them. By Green's theorem the definition (2.19) is equivalent to

$$W = \int_D (\boldsymbol{x}\cdot\nabla\phi + 3\phi)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 - \int_\infty \phi(\boldsymbol{x}\cdot\boldsymbol{n})\,\mathrm{d}s.$$

Here as before the normal vector $\boldsymbol{n}$ is outward on the surface at infinity. Hence a straightforward calculation shows that

$$\frac{\mathrm{d}W}{\mathrm{d}t} = 5K - 2\sigma|S| + 3P\mathscr{V}. \tag{2.20}$$

(No contribution is made by an integral at infinity because $\phi_t + p = O(1/r^4)$ there.) Note that this result is consistent with a state of rest; for then $K = 0$, $\mathscr{V} = \frac{4}{3}\pi a^3$, $|S| = 4\pi a^2$ and $P = 2\sigma/a$.

An interesting implication of (2.20) concerns infinitesimal shape oscillations of a bubble that are superposed on its spherical form at rest (cf. Lamb 1932, §275). Changes of pressure in the contained gas are of second-order smallness; and, as is well known, the mean kinetic energy $\overline{K}$ of such simple-harmonic motions equals their mean potential energy. Hence, taking as the model that $\mathscr{V}$ is fixed, one deduces from (2.20) that owing to the oscillations the mean pressure inside the bubble changes by an amount

$$\overline{P} - \frac{2\sigma}{a} = -\frac{\overline{K}}{\mathscr{V}} < 0. \tag{2.21}$$

On the alternative supposition that P is related to by the adiabatic law $P\mathscr{V}^\gamma =$ const. with $\gamma \geqslant 1$, allowance has to be made for changes in mean radius which affect both the second and third terms on the right of (2.20). A simple calculation leads to

$$\overline{P} - \frac{2\sigma}{a} = -\frac{\{3\gamma/(3\gamma-1)\}\overline{K}}{\mathscr{V}}. \tag{2.22}$$

As might be expected, the previous result is recovered in the limit $\gamma \to \infty$. In the case $\gamma = 1$, which corresponds to an isothermal gas law, the numerical factor on the right of (2.22) is $\frac{3}{2}$ and the reduction in pressure is then the largest possible.

2.5. *The case of fixed volume*

An expedient theoretical model is provided by specifying that $\mathscr{V} =$ const. The model has been studied previously by Moore (1965), El Sawi (1974) and others, and it is justified as a good approximation when, as is usual, hydrodynamic and capillary pressures are much smaller than those needed to produce significant changes in the volume of gas contained by a bubble.

Two more or less equivalent ways of treating it by Hamiltonian theory may be appreciated. In either, $E_0 = K + \sigma|S|$. First, the Hamiltonian is taken to be E_0 but the configuration space for $\boldsymbol{X}$ is specifically delimited by the volume constraint, so that the operation grad has to be interpreted accordingly. Thus, grad E in (2.7) is replaced by grad $E_0 - P$ grad $\mathscr{V}$, where P is the scalar Lagrange multiplier depending on t alone. In effect the modification realigns the vector grad E_0 to lie in the narrowed configuration space. The arguments underlying the relevant version of Noether's theorem need a corresponding adjustment, but the outcome is of course much the same as already summarized.

Alternatively, the Hamiltonian is taken to be $\mathscr{H} = E_0 - P\mathscr{V}$ and the theory proceeds as before; in particular $\mathscr{H}$ replaces E in (2.7). Since $\mathscr{H}$ is explicitly t-dependent through P, the formula (2.10) referred to the modified version of (2.7) gives

$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}t} = -\mathscr{V}\frac{\mathrm{d}P}{\mathrm{d}t},$$

which confirms that $E_0 =$ const.

2.6. *Steady translational motions*

Suppose the movement of the bubble to be a uniform translation in the x_1-direction. Then $\partial \boldsymbol{X}/\partial t = [c, 0, 0]$. Since $\boldsymbol{\Phi}$ too is reckoned as a function of α, β and t, we also have $\partial \boldsymbol{\Phi}/\partial t = 0$ in this case. Thus, when account is still taken of potential energy due to the gas contents, (2.7) reduces to

$$\operatorname{grad} E = \boldsymbol{M}[c, 0, 0, 0]^{\mathrm{T}} = c \operatorname{grad} I_1.$$

But obviously $\mathscr{V} = \text{const.}$ during the steady motion, and so the formulation explained in the preceding paragraph becomes more helpful. We thus arrive at

$$\operatorname{grad} \mathscr{H} = \operatorname{grad} E_0 - P \operatorname{grad} \mathscr{V} = c \operatorname{grad} I_1, \tag{2.23}$$

in which $E_0 = K + \sigma|S|$. Conversely, it is easy to verify that the system of equations (2.23) is only satisfied by a solution $\boldsymbol{U}$ representing a steady motion. Needless to say, a corresponding system of equations applies to steady motions in any other direction.

The system (2.23) constitutes the Euler–Lagrange necessary condition for extremal values of E_0 corresponding to given values of $\mathscr{V}$ and I_1. The internal pressure P and velocity c of the bubble are thus presented as Lagrange multipliers. The variational principle mainly in view, which will be considered further at the end of §3, can be stated as follows. Given an I_1 (not too large; see below), the motion realizing the *minimum* of E_0 is a steady translation. In application of the principle, competitors for the minimum can be delimited to motions symmetric about the x_1-axis, and when I_1 is not too large it can be expected moreover that the absolute minimum will be realized in the class of axisymmetric motions. But there is evidence that this attribution is not always valid (see §5.2). The present variational characterization is the counterpart of known results for steady water waves (Benjamin 1972, Appendix; 1974, §2).

Since according to the constant-volume model E_0, I_1 and $\mathscr{V}$ are all constants of any free motion, the characterization of a steady motion as conditional minimizer of E_0 carries an implication of stability. Specifically, the non-negative invariant $E_0 - \min E_0$ may serve as a Lyapunov function on which to base a proof of stability. In the definition of a useful metric, translations of the bubble would have to be factored out somehow, and a proof of stability in respect of shape would also be quite complicated otherwise (cf. Benjamin 1972). But the minimizing property is at least a persuasive pointer to the likelihood of stability.

The qualification that I_1 should not be too large for a given $\mathscr{V}$ is necessary because of the basic assumption that $\varDelta$ is simply connected. A large impulse can be realized by the motion of a bubble with small $\mathscr{V}$ only if the bubble becomes toroidal. This point has been discussed fully by Benjamin & Ellis (1967). Extensions of the theory dealing with multiply connected bubbles will be outlined in § 5.

2.7. *Steady spinning or spiralling motions*

This general class of bubble motions will now be shown to admit a comprehensive variational characterization, identified here for the first time. Suppose that a bubble maintains its shape while translating with constant velocity c in the x_1-direction and also rotating with constant angular velocity ω about the x_1-axis. Two possibilities are in view: either the centroid of the bubble remains on the x_1-axis, in which case the bubble is said to have a spinning motion, or the centroid moves in a helical path around the x_1-axis and so the bubble has a spiralling motion. In either case we have

$$\frac{\partial \boldsymbol{X}}{\partial t} = c[1,0,0]^{\mathrm{T}} + \omega[0, -X_3, X_2]^{\mathrm{T}},$$

and in the present formulation $\partial\Phi/\partial t = 0$ (i.e. the evaluation Φ of ϕ at the bubble surface evidently translates and rotates in step with the shape, therefore depending only on α and β). Hence, in the light of our finding (2.14) about L_1 in relation to the rotation group, the equations of motion are seen to reduce in this case to

$$\operatorname{grad} E_0 - P \operatorname{grad} \mathscr{V} = c \operatorname{grad} I_1 + \omega \operatorname{grad} L_1. \tag{2.24}$$

This system of equations constitutes the Euler–Lagrange necessary conditions for extremal values of E_0 corresponding to given values of $\mathscr{V}$, I_1 and L_1. In other words, when the volume $\mathscr{V}$ of the bubble and the impulsive wrench composed of I_1 and L_1 are prescribed, a shape realizing a stationary value of total energy determines a bubble that performs a steady spinning or spiralling motion. Since E_0, I_1 and L_1 are all constants of any free motion, a conditional minimum of E_0 again indicates the likelihood of the steady motion so characterized being stable in respect of bubble shape.

For steady spinning motions a different but equivalent formulation will be noted in §3.1. The simplest case arises when $I_1 = 0$ and E_0 is minimized for given $\mathscr{V}$ and $L_1 \neq 0$. Then the bubble rotates about the x_1-axis without translating. Analysis is particularly easy when L_1 is infinitesimal, specifically $O(\epsilon^2)$ if ϵ is the amplitude of the perturbation from spherical form suffered by the spinning bubble. The outcome of the present theory can then be anticipated from well-known results, and details of the confirmatory derivation from (2.24) can be passed over here. Considering vibrations of a nearly spherical bubble of mean radius a, Lamb (1932, p. 475) showed in general that their frequency ω is given by

$$\omega^2 = (n-1)(n+1)(n+2)\frac{\sigma}{\rho a^3}, \tag{2.25}$$

where n is the order of the spherical harmonic (S_n in Lamb's notation) describing the radial displacements of the bubble surface. (Note that dependence on the density ρ of the surrounding liquid, hitherto taken as 1 for convenience, is suitably made explicit in the preceding formula.) For the present application we need sectoral harmonics of order $n \geqslant 2$. Thus, in terms of spherical coordinates (r, θ, ψ), the surface of the bubble is described by

$$r = a + \epsilon P_n^n(\cos\theta)\cos n(\psi - \omega t).$$

It may readily be confirmed that the minimum of E_0 for given $L_1 \neq 0$ is realized with $n = 2$. Then, as $P_2^2(\cos\theta) = \frac{3}{2}(1-\cos 2\theta)$, the bubble is an ellipsoid with principal diameters $2a$ and $2a \pm 6\epsilon$ spinning with an angular velocity ω that is shown by the preceding formula to satisfy

$$\omega^2 = \frac{12\sigma}{\rho a^3}. \tag{2.26}$$

The spheroidal forms corresponding to $n = 3, 4, \ldots$ spin at successively higher angular velocities and their motions realize higher stationary values of E_0 for given L_1.

Steadily spiralling motions of bubbles are governed by the same basic equations (2.24) as spinning ones, but are harder to analyse in detail. They will be explained by an approximate treatment in §5, where interesting questions about the occurrence of such motions in practice will be addressed finally.

3. Restricted theory

A simplification of the Hamiltonian theory becomes available on the assumption that the surface S of the bubble can be represented non-parametrically in terms of an orthogonal coordinate system, for example spherical coordinates. Let α and β denote two of the coordinates, to be treated in the same way as the parameters given this notation in §2 so that γ_1 and J have the same definitions as before; and let ξ denote the third. It is understood that (ξ, α, β) is a right-handed system. The needed assumption is that S can be described by an equation

$$\xi = \Xi(\alpha, \beta, t), \tag{3.1}$$

in which Ξ is a single-valued function of α and β.

Given any initial state of a bubble, such a representation of S can always be found which will remain applicable for a finite time; however, its scope will be limited. For example, a translating bubble will after a comparatively short time pass beyond the range of spherical coordinates that may apply at first. Again, in order to make cylindrical polar coordinates applicable, the motions are limited to those such that S continues to intersect the z-axis orthogonally. In that application, a few details of which will be noted later, the interval for $\beta \equiv z$ is generally time-dependent, say $[a(t), b(t)]$; but this feature presents no difficulty.

Whenever (3.1) holds, whatever the choice of orthogonal coordinate system, the dynamical problem is reducible to an example of Hamilton's equations in canonical form. This result may be deduced directly from the equations of motion coupled with the appropriate definition of variational derivatives; but there is more to be learned by obtaining it as follows from the generalized formulation (2.7). As is standard, let g denote the (necessarily positive) determinant and g_{11} the first diagonal component of the metric tensor for the system (ξ, α, β). In keeping with previous notation, the capitals G and G_{11} will denote the respective evaluations of g and g_{11} at the surface S. Also at S, the direction cosines Λ_i of the coordinate line ξ with respect to the Cartesian axes X_i are given by

$$\Lambda_i = \frac{1}{G_{11}^{\frac{1}{2}}} \left(\frac{\partial x_i}{\partial \xi}\right)_{x=X},$$

from which and from the definition of γ_i at the beginning of §2 it is found that

$$\gamma_i \Lambda_i = \frac{1}{G_{11}^{\frac{1}{2}}} \left[\frac{\partial(x_1, x_2, x_3)}{\partial(\xi, \alpha, \beta)}\right]_{x=X} = \frac{G^{\frac{1}{2}}}{G_{11}^{\frac{1}{2}}}. \tag{3.2}$$

Note also that infinitesimal variations $\dot{X}_i$ in the Cartesian coordinates of points in S are in view of (3.1) expressible by

$$\dot{X}_i = \Lambda_i \dot{\Xi}, \tag{3.3}$$

so that

$$\gamma_i \dot{X}_i = \frac{G^{\frac{1}{2}}}{G_{11}^{\frac{1}{2}}} \dot{\Xi}. \tag{3.4}$$

For the same reason,

$$\gamma_i \frac{\partial X_i}{\partial t} = \frac{G^{\frac{1}{2}}}{G_{11}^{\frac{1}{2}}} \frac{\partial \Xi}{\partial t}. \tag{3.5}$$

In the reformulation based on (3.1), the dependent variables are $\Xi(\alpha,\beta,t)$ and $\Phi(\alpha,\beta,t)$. Variational derivatives of functionals dependent on Ξ and Φ are defined as follows, where the crucial step is to introduce the weighted element $d\mu = (G^{\frac{1}{2}}/G^{\frac{1}{2}}_{11})\,d\alpha\,d\beta$ of the domain Ω for (α,β). Let us use asterisks to denote functional derivatives in the sense adopted originally in §2, and take the absence of asterisks to refer to the present sense. The first variation of any functional f due to variations of Φ, with S held fixed, is alternatively expressed by

$$\dot{f} = \int_\Omega \left(\frac{\delta f}{\delta \Phi}\right)^* \dot{\Phi}\,d\alpha\,d\beta = \int_\Omega \left(\frac{\delta f}{\delta \Phi}\right)\dot{\Phi}\,d\mu;$$

and since $\dot{\Phi}$ is arbitrary it follows that

$$\frac{\delta f}{\delta \Phi} = \frac{G^{\frac{1}{2}}_{11}}{G^{\frac{1}{2}}}\left(\frac{\delta f}{\delta \Phi}\right)^*. \tag{3.6}$$

Correspondingly, for variations in S with Φ fixed, we have

$$\dot{f} = \int_\Omega \left(\frac{\delta f}{\delta X_i}\right)^* \dot{X}_i\,d\alpha\,d\beta = \int_\Omega \left(\frac{\delta f}{\delta \Xi}\right)\dot{\Xi}\,d\mu,$$

whence the substitution of (3.3) for $\dot{X}_i$ and the fact that $\dot{\Xi}$ is arbitrary show that

$$\frac{\delta f}{\delta \Xi} = \frac{G^{\frac{1}{2}}_{11}}{G^{\frac{1}{2}}} A_i \left(\frac{\delta f}{\delta X_i}\right)^*. \tag{3.7}$$

Note from (2.1) and from $(\delta \mathscr{V}/\delta X_i)^* = \gamma_i$ that according to (3.2)

$$\frac{\delta |S|}{\delta \Xi} = 2H, \quad \frac{\delta \mathscr{V}}{\delta \Xi} = 1, \tag{3.8}$$

whatever the choice of the coordinate system (ξ,α,β).

The reduction of (2.7) now proceeds easily in much the same way as (2.7) was shown to be equivalent to (2.4) and (2.5). The fourth row of (2.7) simplifies by virtue of (3.5), (3.6) and cancellation of a common factor $G^{\frac{1}{2}}/G^{\frac{1}{2}}_{11} > 0$. Then one can use this result after multiplying the first three rows by A_i respectively and summing by means of (3.2) and (3.7). The final results are

$$\left.\begin{aligned} \Xi_t &= -\frac{\delta E}{\delta \Phi} = N\Phi_{(n)}, \\ \Phi_t &= \frac{\delta E}{\delta \Xi} = -\tfrac{1}{2}q^2 + 2\sigma H - P + N\Phi_{(n)}\,\Phi_{(\xi)}, \end{aligned}\right\} \tag{3.9}$$

where $N = (JG^{\frac{1}{2}}_{11})/G^{\frac{1}{2}}$. In the case that $\mathscr{V}$ is constant, E is here as before to be replaced by $E_0 - P\mathscr{V}$, with $E_0 = K + \sigma|S|$. Because $\Phi_{(t)} = \Phi_t - \Phi_{(\xi)}\,\Xi_t$, these equations can at once be seen to express, in the coordinate system for (3.1), the kinematic and dynamic boundary conditions at the bubble surface S.

(Note that $N = ds/d\mu$ in (3.9) is a number that generally varies with position on S but everywhere satisfies $N \geqslant 1$. In the case of spherical coordinates (r,θ,ψ), for which the particular form of (3.1) is $r = R(\theta,\psi,t)$, we have $G_{11} = 1$, $G^{\frac{1}{2}} = R^2 \sin\theta$, and hence find $N = \{1 + R^{-2}R_\theta^2 + (R\sin\theta)^{-2}R_\psi^2\}^{\frac{1}{2}}$.)

Equations (3.9) compare with (2.8) in Benjamin & Olver (1982). The difference in signs is due merely to the fact that Ξ is measured towards the liquid, whereas in the previous account dealing with water waves the corresponding variable, the elevation of the free surface, was measured in the opposite direction. The canonical Hamiltonian representation of the analogous water-wave problem was noted first by Zakharov (1968); and the variational properties thus implied have also been explored by Benjamin (1972, Appendix; 1974, §2), Miles (1977), Milder (1977) and others. For an alternative view of the present representation, reference may be made to the paper by Lewis *et al.* (1985) cited in §1.

Associated with the canonical representation there is an obvious Poisson bracket. For any two real functionals f and $\hat{f}$ with variational derivatives as presently defined, the bracket is

$$[f,\hat{f}] = \int_{\Omega}\left\{\frac{\delta f}{\delta\Phi}\frac{\delta\hat{f}}{\delta\Xi}-\frac{\delta f}{\delta\Xi}\frac{\delta\hat{f}}{\delta\Phi}\right\}\mathrm{d}\mu. \tag{3.10}$$

It follows at once from (3.9) that, if f has no explicit dependence on t, then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = [f,E]. \tag{3.11}$$

In particular, f is invariant for any solution of (3.9) if $[f,E]=0$. This property can be used to reconfirm the simpler of the conservation laws demonstrated in §2, and the more complicated ones can be verified by appropriate extensions of the approach. As regards details of the relation between symmetries and conservative properties, however, the present formulation is less straightforward than the generalized one in §2. From (3.9) a Lagrangian least-action principle for bubble motions can also be derived straightforwardly (cf. Benjamin & Olver 1982, p. 168); but since it seems to add nothing useful, this derivation is omitted.

3.1. *Cylindrical coordinates*

Use of these coordinates presents a few special features which deserve attention, although of course the essentials merely exemplify the preceding theory. With negligible risk of confusion the symbols r, θ and R will be re-used in writing (r,θ,z) for the cylindrical system, and for the equation describing S

$$r = R(\theta,z,t). \tag{3.12}$$

We have $G_{11}=1$, $G^{\frac{1}{2}}=R$, so $\mathrm{d}\mu = R\,\mathrm{d}\theta\,\mathrm{d}z$, and $N=(1+R^{-2}R_\theta^2+R_z^2)^{\frac{1}{2}}$. The axial coordinate z is taken to be the same as x_3.

As already mentioned, the domain $\Omega=[0,2\pi]\times[a(t),b(t)]$ is generally t-dependent, and S has to continue intersecting the z-axis orthogonally for (3.12) to remain valid (i.e. for R to remain a *single-valued* function of θ and z). At a pole of S, say the furthest along the axis, it is thus essential that $|R_z|\to\infty$ and so $|N|\to\infty$ as $z\to b$. But since $\mathrm{d}s = N\,\mathrm{d}\mu \sim \mathrm{d}(\frac{1}{2}R^2)\,\mathrm{d}\theta$ as $z\to b$, the integral over Ω expressing $|S|$ is plainly convergent, and likewise other relevant integrals such as that for I_3 given below. For similar reasons, moreover, the first variation of these integrals is in every case found not to depend explicitly on variations of a and b. So the Hamiltonian representation (3.9) of the dynamical problem is borne out unambiguously.

There is only need to consider the axial components of the linear Kelvin impulse $\boldsymbol{I}$ and the impulsive couple $\boldsymbol{L}$. Because $\mathrm{d}s = N\,\mathrm{d}\mu$ and $N\boldsymbol{n} = (1, R^{-1}R_\theta, -R_z)$, they are given by

$$I_3 = -\int_S \Phi n_3\,\mathrm{d}s = \int_b^a\int_0^{2\pi}\Phi R_z R\,\mathrm{d}\theta\,\mathrm{d}z,$$

$$L_3 = -\int_S \Phi(\boldsymbol{X}\times\boldsymbol{n})_3\,\mathrm{d}_s = \int_a^b\int_0^{2\pi}\Phi R_\theta R\,\mathrm{d}\theta\,\mathrm{d}z.$$

It follows that

$$\frac{\delta I_3}{\delta R} = -\Phi_z, \quad \frac{\delta I_3}{\delta \Phi} = R_z, \tag{3.13}$$

$$\frac{\delta L_3}{\delta R} = -\Phi_\theta, \quad \frac{\delta L_3}{\delta \Phi} = R_\theta. \tag{3.14}$$

These expressions confirm the relation between impulse and spatial symmetries, as demonstrated more generally in §2. The infinitesimal generators of the axial translation and axial rotation symmetry groups for (3.9) are ∂_z and ∂_θ respectively, and their representatives in the sense associated with the symbol V in §2 are respectively $V = -U_z$ and $V = -U_\theta$, where $U = [R, \Phi]^{\mathrm{T}}$ is any solution of (3.9). Thus each case bears out the rule (cf. Benjamin & Olver 1982, §5; Olver 1980, 1983)

$$V = \mathscr{T} \operatorname{grad} f(U),$$

where f stands in turn for the invariant functionals I_3 and L_3, and where $\mathscr{T}$ is the skew-symmetric matrix implied when the system (3.9) is expressed by

$$U_t = \mathscr{T} \operatorname{grad} E(U). \tag{3.15}$$

The variational characterizations of steady motions are hence recovered in a manner superficially different but in fact equivalent to that in §2. For *axisymmetric* forms of S and correspondingly θ-independent Φ (so that $L_3 = 0$), consider the variational principle $E_0 = $ *min. for given values of* $\mathscr{V} > 0$ *and* $I_3 \neq 0$. In view of (3.9) and (3.13), the Euler–Lagrange equations necessarily satisfied by a conditional minimizer are

$$\left.\begin{aligned} -c\Phi_z + P &= \frac{\delta E_0}{\delta R}, \\ cR_z &= \frac{\delta E_0}{\delta \Phi}, \end{aligned}\right\} \tag{3.16}$$

in which c and P are Lagrange multipliers. Comparison with (3.9) shows these equations to be the boundary conditions for an axisymmetric bubble steadily translating with velocity c in the z-direction; for then $R = R(z-ct)$ so that $R_t = -cR_z$, and similarly $\Phi_t = -c\Phi_z$. Note that upon multiplying the second equation by $R\Phi$, integrating over Ω and appealing to Green's theorem, one concludes that for any such steady motion

$$cI_3 = 2K. \tag{3.17}$$

This property is well known (cf. Kochin *et al.* 1964, §7.7) and is easily confirmed to be true for any body moving without rotation in the x_1-direction. It implies, as is otherwise obvious, that c has the same sign as I_3.

Steadily spinning motions are described by the variational principle $E_0 = $ *min. for given* $\mathscr{V} > 0$, I_3 *and* $Q_3 \neq 0$. In view now of (3.14) as well as (3.9) and (3.13), the Euler–Lagrange equations are

$$\left.\begin{aligned} -c\Phi_z - \omega\Phi_\theta + P &= \frac{\delta E_0}{\delta R}, \\ cR_z + \omega R_\theta &= \frac{\delta E_0}{\delta \Phi}, \end{aligned}\right\} \tag{3.18}$$

to which the boundary conditions reduce when $R = R(z-ct, \theta-\omega t)$ and similarly for Φ. Thus the Lagrange multiplier ω respecting the prescribed value of L_3 is the angular velocity of the spiralling motion represented by the conditional minimizer. Generalizing (3.17) we now have

$$cI_3 + \omega L_3 = 2K. \tag{3.19}$$

In the case $I_3 = 0$, the minimizer of E_0 for given $\mathscr{V}$ and $L_3 \neq 0$ represents a motion where a non-axisymmetric bubble spins about the z-axis without moving along it.

It deserves re-emphasis that these conclusions are meaningful only if $|I_3|$ and $|L_3|$ are not too large, so that the existence of conditional minimizers in the considered class of motions may be presumed. Minimax conditional extremals if they exist also represent steady motions, of course, but the motions so characterized are likely to be unstable. (For example, this is so for the infinitesimal spinning motions with $n > 2$ noted at the end of §2.7.) The more serious limitation on the theory is that if $|I_3|$ is large enough for a given $\mathscr{V}$ there is no simply connected bubble that moves steadily.

4. Estimates for axisymmetric steady motions

By use of the simple relation (3.17) particularized to translating axisymmetric bodies with non-varying shape, the theory can be simplified further and enables the dimensionless form of c^2 (the Weber number) to be estimated by a Rayleigh–Ritz procedure. For such a body it is customary to define an added-mass coefficient μ by

$$K = \tfrac{1}{2}\rho\mathscr{V}c^2\mu,$$

corresponding to which

$$I_3 = \rho\mathscr{V}c\mu.$$

(Dependence on the density ρ of the liquid, previously taken to be unity, is here suitably made explicit.) A sphere is well known to have $\mu = \frac{1}{2}$, and for oblate bodies $\mu > \frac{1}{2}$. Similarly, taking $\mathscr{V} = \frac{4}{3}\pi r_e^3$ to define the equivalent spherical radius r_e, we may express the superficial energy of a bubble in terms of a dimensionless coefficient λ, thus

$$\sigma|S| = 4\pi r_e^2\sigma\lambda.$$

A sphere has $\lambda = 1$, which value is of course the absolute minimum. In the class of all axisymmetric surfaces S closing simply connected bubbles, both μ and λ depend only on shape, not on size.

Since a minimizer of $E_0 = K + \sigma|S|$ for a given $\mathscr{V}$ and I_3 automatically satisfies (3.17), it also minimizes

$$F = \frac{I_3^2}{2\rho\mathscr{V}\mu} + 4\pi r_e^2\sigma\lambda. \tag{4.1}$$

Any stationary value of (4.1) requires that

$$0 = \dot{F} = -\frac{I_3^2}{2\rho\mathscr{V}\mu^2}\dot{\mu} + 4\pi r_e^2\sigma\dot{\lambda}$$

$$= -\tfrac{1}{2}\rho\mathscr{V}c^2\dot{\mu} + 4\pi r_e^2\sigma\dot{\lambda};$$

and in terms of the Weber number $W = 2\rho r_e c^2/\sigma$ this condition is

$$W\dot{\mu} = 12\dot{\lambda}. \tag{4.2}$$

It is essential moreover that W be positive, and thus steady translations are impossible for prolate forms of the bubble with $\mu < \frac{1}{2}$. Specifically, (4.2) is satisfied by every infinitesimal perturbation relative to an extremum; but for a prolate form realizing a stationary value of F, there would necessarily be perturbations towards spherical form such that $\dot{\lambda} < 0$ yet $\dot{\mu} > 0$. The fact thus spotlighted is otherwise evident, because a steadily translating bubble must have greater mean curvature at its equator than at its poles for hydrodynamic pressures on S to be balanced by the action of surface tension.

Equation (4.2) shows that a minimum of E_0 for given $\mathscr{V}$ and I_3 is realized by the solution to either of the variational problems

$$\text{(A)}\quad \lambda = \min \quad \text{for given } \mu > \tfrac{1}{2},$$

$$\text{(B)}\quad \mu = \max \quad \text{for given } \lambda > 1.$$

So they recover the original characterization of steady motions and present the main physical parameter W as Lagrange multiplier. Either (A) or (B) is a suitable setting for a Rayleigh–Ritz method of estimating W.

2.1. *Spheroidal approximations*

Covering the one-parameter class of *oblate* ellipsoids of revolution, an expression for μ is available from Lamb (1932, §§114, 373) or Milne-Thomson (1968, §16.54). In terms of $\eta = \operatorname{arc\,sec}(b/a)$, where $b/a \geqslant 1$ is the ratio of the axes, write

$$\mathrm{s} = \sin\eta, \quad \mathrm{c} = \cos\eta = \frac{a}{b}, \quad \tau = \tan\eta.$$

Then

$$\mu = \frac{\tau-\eta}{\eta-\mathrm{sc}}. \tag{4.3}$$

An expression for the surface area of an oblate spheroid is well known, leading to

$$12\lambda = \frac{12|S|}{4\pi r_e^2} = \frac{3}{(b^2a)^{\frac{2}{3}}}\left\{2b^2 + \frac{a^2}{\mathrm{s}}\ln\left(\frac{1+\mathrm{s}}{1-\mathrm{s}}\right)\right\}$$

$$= 6\mathrm{c}^{-\frac{2}{3}} + \frac{3\mathrm{c}^{\frac{4}{3}}}{\mathrm{s}}\ln\left(\frac{1+\mathrm{s}}{1-\mathrm{s}}\right). \tag{4.4}$$

According to (4.2), the best spheroidal approximation to solutions of the problems (A) or (B) makes

$$W = \frac{12\lambda'(\eta)}{\mu'(\eta)}.$$

Hence differentiation of (4.3) and (4.4) leads directly to

$$W = \frac{\mathrm{c}^{\frac{1}{3}}(\eta-\mathrm{sc})^2}{\mathrm{s}^3[\eta(3+\tau^2)-3\tau]}\left\{6+4\tau^2-\left(\frac{4-\mathrm{c}^2}{\mathrm{s}}\right)\ln\left(\frac{1+\mathrm{s}}{1-\mathrm{s}}\right)\right\}. \tag{4.5}$$

Thus W is estimated explicitly as a function of η, and correspondingly of b/a.

Because the Rayleigh principle is exemplified by its variational basis, the result (4.5) can be expected to give a quite close approximation to W, particularly for values of b/a not much larger than 1. Figure 2 presents graphs of W and μ against b/a, from (4.5) and (4.3) respectively, also a graph of dimensionless impulse $\mu W^{\frac{1}{2}}$. An interesting feature is that W has a maximum value 3.271 at $b/a = 3.722$, $\mu = 2.20$, which estimate was previously obtained by El Sawi (1974, figure 2). A similar but less accurate graph of W against b/a had been found by Moore (1965, figure 1) upon assuming spheroidal bubbles and satisfying the boundary conditions only at the poles

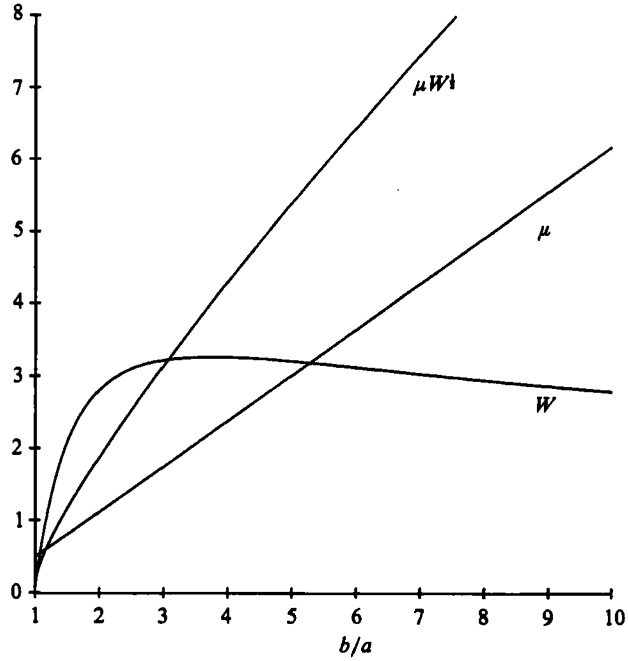

FIGURE 2. Calculated Weber number W, inertial coefficient μ and dimensionless impulse $\mu W^{\frac{1}{2}}$ plotted against axis ratio $b/a = \sec\eta$ for oblate spheroidal bubbles.

and equator. Approximate findings such as the present can be checked by comparison with results computed numerically by Miksis *et al.* (1981, figure 5), whose estimate is that $W_{\max} = 3.23$ at $b/a = 3.85$. The curve W *vs.* b/a obtained by their computations is very close to that in figure 2 as far as values of b/a more than twice the optimum, even though for such values the computed shapes of bubbles differs significantly from spheroidal. Other relevant numerical results, bearing particularly on the dependability of our inviscid-fluid model, are available from the study by Ryskin & Leal (1984, §4.2). Their computations included viscous effects on axisymmetric bubbles and confirmed that the shapes of bubbles depends predominantly on W when the Reynolds number for them is as high as 100 or 200.

It is remarkable that the present simple estimate of the maximum reached by W is only 3 % greater than the critical value estimated experimentally by Hartunian & Sears (1957), above which value the rectilinear motion of bubbles in pure liquids with small viscosity becomes unstable. A reasonable conjecture is that the maximum of W is associated with bifurcation from the class of axisymmetric conditional extrema into non-axisymmetric ones (see §5.2).

Despite a quite different formulation, the result (4.5) is in fact equivalent to the result given by El Sawi (1974, equation (3.12)), which he derived by 'lengthy and tedious computations' based on a virial method. The agreement is hardly surprising: his approach, like the present one, serves to locate the best fit to the bubble problem in the class of spheroidal shapes. But the present derivation of (4.5) appears vastly simpler than El Sawi's, both conceptually and in the details of the computations.

5. Spiralling bubbles

Steady helical motions of axisymmetric *solid* bodies in an infinite liquid are well understood from an analysis due to Kirchoff which was reported by Lamb (1932, §129; see also Ramsey 1935, §8.56). For a solid body with angular inertia about its axis of symmetry, an infinite range of such motions is possible corresponding to continuously disposable values of a parameter k with the dimension of length (Lamb, 1932, p. 179, (2) and (3)). But for an axisymmetric bubble, which of course has no angular inertia about its axis of symmetry, the same analysis shows that for a spiralling motion the shape of the bubble must be oblate and k is limited to a single value. Although these and other necessary properties of spiralling bubbles can be deduced from the original analysis, it will be interesting to use the variational characterization (2.24) in order to derive a complete description including that of shape.

Let us assume the bubble in steady motion to be axisymmetric, and take moving axes (x, y, z) with x along the axis of symmetry. Because of the symmetry and the absence of angular inertia about the x-axis, the z-axis can obviously be chosen to intersect the fixed axis, say x_1, relative to which the impulsive wrench composed of I_1 and L_1 is taken to be prescribed. It will save space to assume that the axes z and x_1 are orthogonal and that the linear and angular velocities of the bubble have no component in the z-direction. (Otherwise allowance can be made for a variable angle between these axes and for non-zero values of these components: then the extension of (5.4) to include the three additional variables straightforwardly confirms the present assumptions.) Accordingly, respective to (x, y, z), the linear velocity of the bubble is expressed by $(u, v, 0)$ and its angular velocity by $(p, q, 0)$, although of course the component p has no dynamical significance. Let R denote the radial distance from the x_1-axis to the centre of the bubble, and θ the angle between the direction of this axis and that of the bubble's axis of symmetry, as shown in figure 3.

On the further assumption that the bubble is symmetric about the (y, z)-plane (which assumption can readily be justified *a posteriori*), the kinetic energy K of the fluid motion is given by

$$2K = Au^2 + Bv^2 + Qq^2, \tag{5.1}$$

where A, B and Q are inertial coefficients calculable in principle for any shape with the supposed symmetries (cf. Lamb 1932, p. 173). Also, with reference to figure 3, the components of the impulsive wrench (I_1, L_1) are seen to be in general

$$I_1 = Au\cos\theta + Bv\sin\theta, \tag{5.2}$$

$$L_1 = R(-Au\sin\theta + Bv\cos\theta) + Qq\sin\theta. \tag{5.3}$$

Appealing now to the variational principle (2.24) for steady motions, we have that

$$\frac{\partial}{\partial U}(K - cI_1 - \omega L_1) = 0, \tag{5.4}$$

where U represents any of the parameters u, v, q, R and θ, each of which can be varied independently of bubble size and shape. With U as u, v and q, (5.4) shows respectively that

$$\left.\begin{aligned} u &= c\cos\theta - \omega R\sin\theta, \\ v &= c\sin\theta + \omega R\cos\theta, \\ q &= \omega\sin\theta. \end{aligned}\right\} \tag{5.5}$$

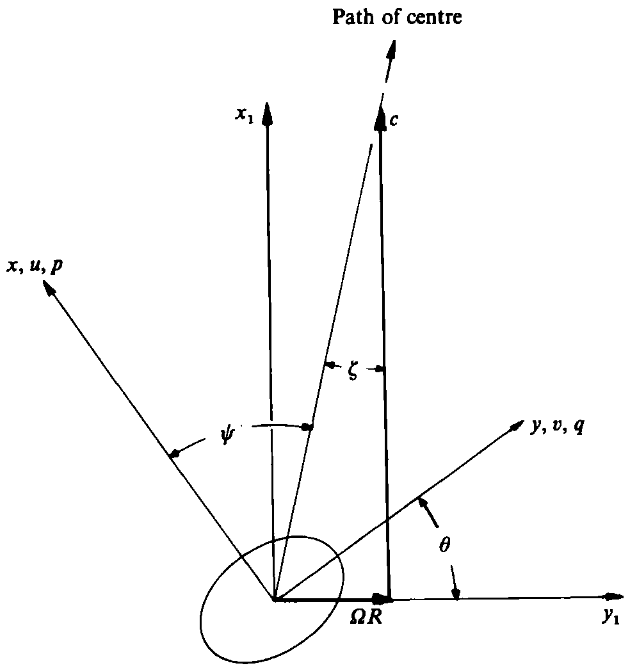

FIGURE 3. Diagram defining angles θ, ψ and ζ.

These three kinematic identities simply show that the centre of the bubble moves steadily along a helical path with radius R and pitch $2\pi c/\omega$. The whole bubble translates with velocity c in the direction of the x_1 axis while rotating about this axis with angular velocity ω.

We next have

$$\frac{\partial}{\partial R}(K - cI_1 - \omega L_1) = \frac{\partial}{\partial R}(-\omega L_1) = 0,$$

which implies that

$$-Au \sin\theta + Bv \cos\theta = 0 \tag{5.6}$$

if neither ω nor R is zero. As can be expected, (5.6) merely confirms that in the steady motion there is no component of linear impulse other than in the x_1-direction. Furthermore, (5.2) and (5.3) reduce by virtue of (5.6) to

$$I_1 = Au \sec\theta \tag{5.7}$$

and

$$L_1 = Qq \sin\theta = \frac{Qq^2}{\omega}. \tag{5.8}$$

The last of the necessary conditions (5.4) for steady motion gives

$$0 = \frac{\partial}{\partial\theta}(cI_1 + \omega L_1) = c(-Au \sin\theta + Bv \cos\theta)$$
$$+\omega\{-R(Au \cos\theta + Bv \sin\theta) + Qq \cos\theta\},$$

whence in view of (5.6), (5.7) and (5.8) it follows that

$$I_1 R = L_1 \cot\theta. \tag{5.9}$$

From (5.9), substituting (5.7), (5.8) and $\omega R = v\cos\theta - u\sin\theta = \{1-(B/A)\}v\cos\theta$, one obtains

$$(A-B)\,uv = Qq^2\cot\theta,$$

from which elimination of u by further use of (5.6) leads to

$$\frac{v^2}{q^2} = \frac{AQ}{B(A-B)}. \tag{5.10}$$

This quotient is the same as k^2 in the aforementioned account by Lamb (1932, §129). Note that (5.10) implies $A > B$: that is, an axisymmetric bubble in steady spiralling motion must be oblate.

Attributes of the helical path followed by the centre of the bubble are deducible as follows from the preceding equations. Write $h = A/B$, remembering that $h > 1$ by (5.10), and let ψ denote the angle of yaw between the path and the x-axis, as shown in figure 3. Then (5.6) gives

$$\tan\psi = \frac{v}{u} = h\tan\theta. \tag{5.11}$$

Expressing v/u alternatively by (5.5), we hence obtain upon rearrangement

$$\cot\zeta = \frac{c}{\omega R} = \frac{\cot\theta + h\tan\theta}{h-1} = \frac{\tan\psi + h\cot\psi}{h-1}. \tag{5.12}$$

Here $\zeta = \psi - \theta$ is the angle between the path and the direction of the x_1-axis, as also shown in figure 3. It can be presumed that $0 < \zeta < \frac{1}{2}\pi$ (i.e. both c and ω have positive values). So in accord with (5.11) and (5.12) we have

$$0 < \theta < \psi < \tfrac{1}{2}\pi.$$

To compare ζ with θ, (5.12) can be rearranged to give the inequalities

$$(h-1)(T-hT^3) < \tan\zeta < (h-1)(T-hT^3+h^3T^5),$$

where $T = \tan\theta$. These show that when the bubble is nearly spherical, so that $0 < (h-1) \ll 1$, and θ is small, then ζ is very close to $(h-1)\theta$ and so much smaller than θ. Note also that both expressions to the right in (5.12) have the absolute minimum value $2h^{\frac{1}{2}}/(h-1)$. So, irrespective of θ, we have

$$0 < \tan\zeta < \frac{1-h}{2h^{\frac{1}{2}}}.$$

Substitution of (5.5) into (5.10) and use of (5.12) lead to

$$R = \left[\frac{(h-1)\,Q}{A}\right]^{\frac{1}{2}} \sin\theta\cos\theta, \tag{5.13}$$

whence further use of (5.12) shows the pitch of the spiral path to be given by

$$\frac{2\pi c}{\omega} = 2\pi R\cot\zeta = 2\pi\left[\frac{Q}{(h-1)\,A}\right]^{\frac{1}{2}}(\cos^2\theta + h\sin^2\theta). \tag{5.14}$$

The extreme smallness of $\tan\zeta = \omega R/c$ noted above in the case of small $(h-1)$ and θ is thus illuminated. For a bubble of unit equivalent spherical radius, since $Q = O[(h-1)^2]$ as $(h-1)\to 0$, we have

$$R = O[(h-1)^{\frac{3}{2}}\theta] \quad \text{and} \quad \frac{c}{\omega} = O[(h-1)^{\frac{1}{2}}].$$

Note that (5.13) and (5.14) determine the helical trajectory completely in terms of the bubble's three inertial coefficients and either θ or the yaw angle ψ according to (5.11).

It remains to determine the shape of spiralling bubbles. We could proceed as at the start of §4, explicitly expressing conditional variations of $K+\sigma|S|$ for fixed $\mathscr{V}$ and given I_1 and L_1 as parameters; but the outcome can be inferred more directly from (2.24). Let $\dot{A}$, $\dot{B}$ and $\dot{Q}$ denote variations of A, B and Q corresponding to infinitesimal variations of the bubble surface S that leave the volume $\mathscr{V}$ unchanged but are otherwise arbitrary. In the light of the kinematical conditions (5.5) it appears at once from (5.1) and (5.2) that, for variations in this class,

$$\dot{K} = \tfrac{1}{2}\dot{A}u^2+\tfrac{1}{2}\dot{B}v^2+\tfrac{1}{2}\dot{Q}q^2 = \tfrac{1}{2}(c\dot{I}_1+\dot{L}_1).$$

Hence, as $\dot{\mathscr{V}} = 0$ for these variations, (2.24) shows that

$$\tfrac{1}{2}\dot{A}u^2+\tfrac{1}{2}\dot{B}v^2+\tfrac{1}{2}\dot{Q}q^2 = \sigma|\dot{S}|. \tag{5.15}$$

A complete characterization of bubble shapes is provided in that (5.15) must be satisfied for arbitrary variations in the considered class, and approximations may be found by a Rayleigh–Ritz procedure based on this variational principle.

From (5.15), expressing q^2 as a multiple of v^2 by use of (5.10) and v^2 as a multiple of u^2 by use of (5.6), one obtains

$$u^2\left[\dot{A}+(A\tan^2\theta)\left\{h\frac{\dot{B}}{B}+(h-1)\frac{\dot{Q}}{Q}\right\}\right] = 2\sigma|\dot{S}|. \tag{5.16}$$

Next u^2 can be expressed as a multiple of c^2, where c is the velocity of the bubble in the x_1-direction, so being given by

$$c = u\cos\theta+v\sin\theta = u\left\{\cos\theta+\frac{h\sin^2\theta}{\cos\theta}\right\}.$$

Hence, upon the introduction of dimensionless inertial coefficients μ, ν and ξ defined for given $\mathscr{V} = \frac{4}{3}\pi r_e^3$ by

$$A = \rho\mathscr{V}\mu, \quad B = \rho\mathscr{V}\nu, \quad Q = \rho\mathscr{V}r_e^2\xi,$$

together with the surface-area coefficient λ and the Weber number W as defined in §4 and now based on c, the result is

$$\frac{W\cos^2\theta}{(\cos^2\theta+h\sin^2\theta)^2}\left[\dot{\mu}+(\mu h\tan^2\theta)\left\{\frac{\dot{\nu}}{\nu}+\left(\frac{h-1}{h}\right)\frac{\dot{\xi}}{\xi}\right\}\right] = 12\dot{\lambda}. \tag{5.17}$$

As it evidently must, this result recovers (4.2) when $\theta = 0$; and $\theta > 0$ can be treated as a parameter of possible spiralling motions. With θ and W given, the bubble shape can in principle be determined fully by solving (5.17), whereupon properties of the respective helical trajectory can be found from (5.13) and (5.14).

5.1. *Spheroidal approximations*

The class of oblate ellipsoids of revolution is now reconsidered in order to obtain estimates for spiralling bubbles. To extend the idea used in §4.1 and find the best spheroidal approximation to solutions of (5.17), the arbitrary variations $\dot{\mu}$, $\dot{\nu}$, $\dot{\xi}$ and $\dot{\lambda}$ are particularized to derivatives with respect to the single parameter $\eta\in[0,\frac{1}{2}\pi]$ covering this class of shapes. Expressions for μ and for λ have already been given as (4.3) and (4.4), while expressions for ν and ξ can be obtained from formulae given in Lamb (1932, §§114, 373), Milne-Thomson (1968, §§17.53, 17.54) or Kochin *et al.* (1964, §7.8). In terms of the notation introduced before (4.3), the required results are

$$\nu = \frac{\eta - \mathrm{sc}}{\tau + \mathrm{s}^2\tau - \eta}, \tag{5.18}$$

$$\xi = \frac{\mathrm{s}^4(\mu-\nu)}{5\mathrm{c}^{\frac{2}{3}}\{\mathrm{s}^2(1+2\mu+\mu\nu)-2(\mu-\nu)\}}. \tag{5.19}$$

It will be informative to consider first the case where the eccentricity s of the elliptic cross-section is small, but where no restriction is placed on θ. From (4.3), (4.4), (5.18) and (5.19) one finds that, as $\mathrm{s}^2 \to 0$,

$$\begin{aligned}
\mu &= \tfrac{1}{2}(1+\tfrac{3}{5}\mathrm{s}^2)+O(\mathrm{s}^4)\\
\lambda &= 1+\tfrac{2}{45}\mathrm{s}^4+O(\mathrm{s}^6),\\
\nu &= \tfrac{1}{2}(1-\tfrac{3}{10}\mathrm{s}^2)+O(\mathrm{s}^4),\\
\xi &= \tfrac{1}{15}\mathrm{s}^4+O(\mathrm{s}^6).
\end{aligned}$$

(Note that, to first order in s^2, one also has $\mathrm{s}^2 = \eta^2 = 2\delta$, where $\delta = (b/a)-1$; and $h-1 = \frac{9}{10}\mathrm{s}^2$.) Hence, to first order with terms $O(\mathrm{s}^4)$ neglected, (5.17) leads directly to

$$W = \frac{32\mathrm{s}^2}{9(1+\frac{3}{2}\sin^2\theta)}. \tag{5.20}$$

With $\theta = 0$, so referring to rectilinear motion of nearly spherical bubbles, this result recovers a formula that has long been known (Moore 1959, (3.5)); and it accounts for the initial slope of the curve in figure 2. As a notable comparison between rectilinear and spiralling motions executed by the same nearly spherical bubble, W and so c^2 are shown by (5.20) to be reduced with increasing θ, down to a minimum of 40 %.

Similarly, to first order in powers of s, equations (5.12), (5.13) and (5.14) lead respectively to

$$\zeta = \frac{9}{10}\mathrm{s}^2 \sin 2\theta, \tag{5.21}$$

$$\frac{R}{r_\mathrm{e}} = \frac{\sqrt{3}}{10}\mathrm{s}^3 \sin 2\theta, \tag{5.22}$$

and

$$\frac{c}{\omega r_\mathrm{e}} = \frac{2}{3\sqrt{3}}\mathrm{s}, \tag{5.23}$$

which bear out the remarks below (5.14). Recalling the definition of W and eliminating c between (5.20) and (5.23), we find that

$$\omega^2 = \frac{12\sigma}{\rho r_\mathrm{e}^3(1+\frac{3}{2}\sin^2\theta)}. \tag{5.24}$$

This result is interesting in comparison with (2.26), to which it reduces as $\theta \to 0$. The equivalence of the two results when θ^2 is small enough to be negligible can be explained by recognizing from (5.21) and (5.22) that ζ and R/r_e are then negligible in a linearized approximation to the motion. Accordingly, the moving surface of the bubble is representable in this approximation by a combination of spherical surface

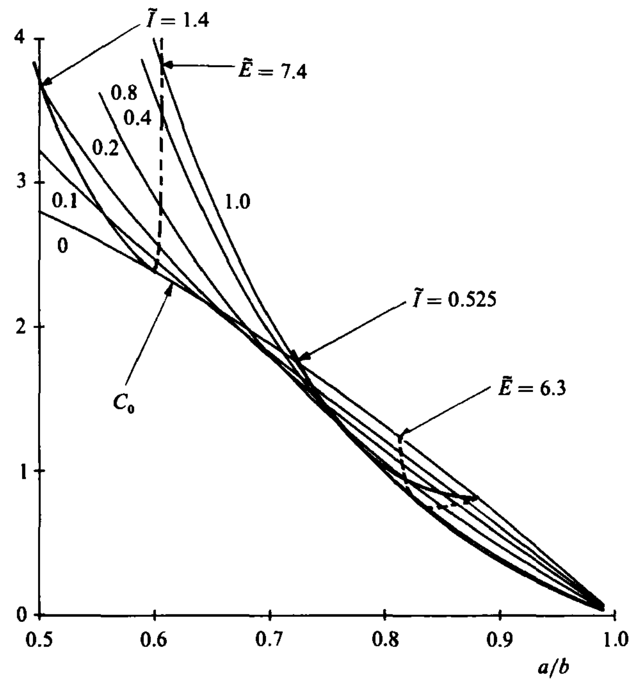

FIGURE 4. Curves W *vs.* a/b in interval [0.5, 1], with parameter $\sin^2\theta$ in interval [0, 1]. Loci $\tilde{I}$ = const. and $\tilde{E}$ = const. are also shown, starting from the same point on curve C_0 for which $\theta = 0$.

harmonics of second order.† Thus Lamb's general formula (2.25) is again applicable with $n = 2$. When θ^2 is not negligible, ω^2 and W, so c^2, are reduced by the same fraction from their values as $\theta \to 0$.

Graphs of W *vs.* a/b for various values of $\sin\theta$ between 0 and 1 are presented in figures 4 and 5. They were computed from (5.17) and the expressions (4.3), (4.4), (5.18) and (5.19) for μ, λ, ν and ξ. An interesting feature prominent in figure 4 is that whereas each curve for $0 < \sin\theta \leqslant 1$ begins on the right by going below the curve for $\theta = 0$, say C_0, it eventually crosses and goes above C_0. The behaviour for small $1-(a/b)^2 = s^2$ is already demonstrated by (5.20), but that for larger $1-(a/b)^2$ is somewhat surprising. Note from figure 4 that the curves below C_0 have an envelope, points on which represent the minimum possible W for the respective a/b (i.e. minimum velocity c for spheroidal bubbles of given volume and shape). The abscissa of the point at which the envelope touches C_0 is seen from (5.17) to be a root of

$$f = \left(2-\frac{1}{h}\right)\frac{\mu'}{\mu} - \frac{\nu'}{\nu} - \left(1-\frac{1}{h}\right)\frac{\xi'}{\xi} = 0, \tag{5.25}$$

† Specifically, changing notation for the moment in order to let (r, θ, ψ) denote spherical coordinates with origin moving with velocity c along the x_1-axis, and writing ϵ for the small angle denoted by θ above, we find that the spheroidal surface S is described by $r_e(1+f)$ with

$$f = -\tfrac{1}{2}\delta\{\tfrac{1}{3} + \cos 2\theta + 2\epsilon \sin 2\theta \cos(\psi - \omega t)\}.$$

The t-independent component, proportional to $P_2(\cos\theta)$, is maintained by the motion in the x_1-direction; and the rotating component, proportional to $P_2^1(\cos\theta)$, is dynamically independent of the other in the present approximation.

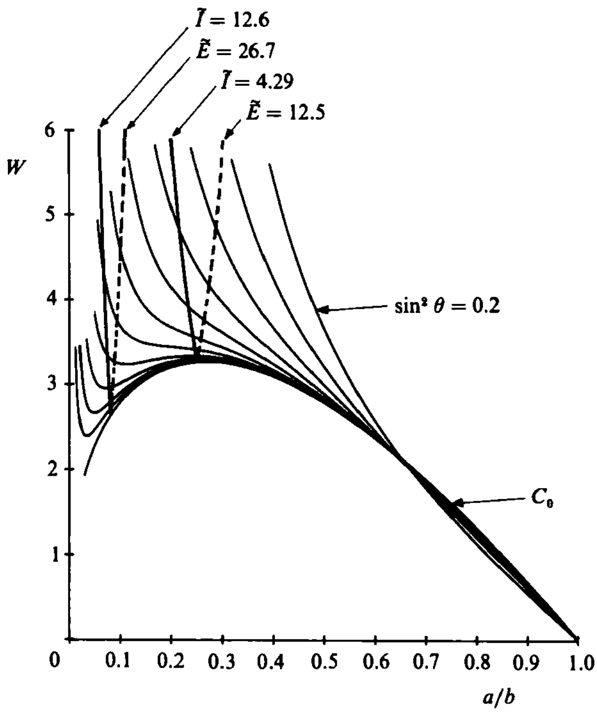

FIGURE 5. Curves W *vs.* a/b with parameter $\sin^2\theta$, whose values are successively halved (thus $\sin^2\theta = 0.1/2^9$ for the curve ending furthest to the left). Two loci $\bar{I} =$ const. and $\tilde{E} =$ const. are also shown, starting from points $a/b = 0.08$ and 0.25 on curve C_0.

because f is proportional to $[\mathrm{d}W/\mathrm{d}\theta^2]_{\theta=0}$ for fixed a/b. Equation (5.25) is found to have only one real root, which is estimated to be $a/b = 0.6234$.

Figure 5 shows that, where a/b is less than this value, the curves depart greatly from C_0 unless $\sin\theta$ is quite small. Those for $0 < \sin^2\theta < 0.1/2^6$, however, are shown to have maxima and then minima to the left of the maximum of C_0 at $a/b = 1/3.722 = 0.2687$.

Figures 4 and 5 also include several pairs of loci $\bar{I} =$ const. and $\tilde{E} =$ const., each pair being started at the same point on C_0. The parameters $\bar{I}$ and $\tilde{E}$ represent impulse and total energy non-dimensionally, being given by

$$\bar{I} = \frac{I_1}{(\frac{8}{9}r_e^5\rho\sigma)^{\frac{1}{2}}} = \frac{\mu W^{\frac{1}{2}}}{\{1+(h-1)\sin^2\theta\}},$$

$$\tilde{E} = \frac{E_0}{(\frac{2}{3}\pi r_e^2\sigma)} = \frac{\mu W\{1+2(h-1)\sin^2\theta\}}{2\{1+2(h-1)\sin^2\theta\}^2} + 6\lambda.$$

In order to interpret these loci, note first that with decreasing a/b along the curve C_0 as far as it is drawn, $\bar{I}$ steadily increases (as the graph of $\mu W^{\frac{1}{2}}$ *vs.* b/a in figure 2 shows), and so too does $\tilde{E}$. Each point on C_0 represents, in the class of spheroidal bubbles with the same volume, a minimum of E_0 for given $I_1 > 0$ and $L_1 = 0$. A locus $\bar{I} =$ const. started on C_0 represents in this class those spiralling motions that realize the same impulse as the purely translational motion represented by the starting point on C_0, and the impulsive couple L_1 in fact steadily increases with distance from C_0 along the locus. It is particularly significant that every curve $\bar{I} =$ const. in figures

4 and 5 is to the *left* of the curve $\tilde{E}$ = const. started at the same point on C_0. Thus each locus $\tilde{I}$ = const. started anywhere along C_0 must intersect loci E = const. starting further to the left along C_0; and in fact $\tilde{E}$ increases steadily with distance from C_0 along each $\tilde{I}$ locus. This property means that, for a given $\mathscr{V}$ and I_1 but without restriction on L_1, a translational motion always realizes the absolute minimum of energy. In other words, the acquisition of impulsive couple always entails a rise in energy. The possibility of a bifurcation into and exchange of stability with the class of spiralling motions is therefore excluded, although it might have been expected to accompany the maximum of C_0 or even the curious reversal of properties at $a/b = 0.6234$.

Although for values of a/b as small as 0.3 the spheroidal approximations lose accuracy in describing the shape of moving bubbles, their success in predicting gross dynamical properties has been noted in §4.1. It seems unlikely that the preceding conclusion would be contradicted by higher approximations.

5.2. *Stability*

By experimental observations on bubbles rising in liquids of small viscosity, Saffman (1956) and Hartunian & Sears (1957) established that steady rectilinear motions become unstable when the velocity of rise exceeds a critical value which depends on the bubble size and physical constants of the liquid. The empirical rule found by Hartunian & Sears has already been noted, namely that there is a critical Weber number $W_c = 3.18$ provided the Reynolds number $2cr_e/\nu$ is greater than about 200. When $W > W_c$ a bubble moves in either a zigzag or helical path, the straight path followed when $W < W_c$ having lost stability. Despite bold attempts at approximate stability analyses by Saffman and by Hartunian & Sears, their theoretical results were largely uncertain, and the main questions of the subject seem to remain unanswered decisively. It is possible that these questions could be settled in large part by a complete analysis based on the Hamiltonian formulation (2.7) or (3.9), and various general properties of continuous Hamiltonian systems might be useful in this connection. But the exact stability problem is evidently not easy, and it will not be tackled here. To conclude the discussion of steady spiralling motions, however, the following remarks about them are worth mention in relation to questions of stability.

In comparison with rectilinear motions of bubbles, spiralling motions have the crucial distinction that the impulsive couple L_1 is non-zero. As noted in §2.4, moreover, L_1 is an invariant of any overall motion in the absence of externally applied forces. Therefore a spiralling motion cannot be evolved from a state of motion with $L_1 = 0$ unless either an external couple acts, such as may arise from pressures on an asymmetric solid boundary fixed nearby, or a complementary motion with impulsive couple $-L_1$ is also evolved somehow and is left behind the spiralling bubble. This basic consideration shows that the emergence of spiralling motions as the outcome of instability must be a very complicated process, being inaccessible to linearized stability theories or other descriptions of that sort. The negative conclusion recorded at the end of §5.1, that spiralling motions are never accountable as I_1-conserving bifurcations from the class of rectilinear motions, is consistent with this appraisal.

Marginally beyond the limit of stability for a straight path, bubbles were observed by Hartunian & Sears (1957) to move most often along helical paths. Saffman (1956) observed zigzag paths to be more usual just beyond the stability limit, and he suggested that the spiralling motion arises from a later instability. Other observations by him appear to identify the true explanation for what happens, to be summarized below. In particular, the zigzag motion was sometimes observed to occur first and then change into the spiralling motion, but never the reverse. Again (Saffman 1956, p. 253) it was found that "a bubble could usually be made to spiral by 'hitting' it, i.e. by placing an obstacle...in the path of the bubble." This observation appears particularly significant in the light of the requirement $L_1 \neq 0$ noted in the preceding paragraph.

It is conjectured that in the neighbourhood of the maximum of W, as calculated accurately by Miksis *et al.* (1981) and reproduced approximately by the curve C_0 in figures 2 or 5, there is a bifurcation into asymmetric shapes of steadily translating bubbles corresponding to given $\mathscr{V}$ and given I_1 above a critical value, but with $L_1 = 0$. These asymmetric bubbles presumably realize a minimum of E_0 smaller than the value of E_0 for the axisymmetric bubble with the same $\mathscr{V}$ and I_1, the latter value having become minimax and so carrying no implication of stability. Exemplifying a common attribute of symmetry-breaking bifurcations near turning points, the branch of asymmetric steady solutions is itself likely to be the centre for time-periodic limit cycles; and such solutions would account for the zigzag motions observed in practice.

No indication of these possibilities is afforded by an approximate analysis on the present lines using only the one-parameter class of oblate spheroidal shapes. But this outcome must be expected since the asymmetric bubbles in question need at least a two-parameter description, because evidently they have to be different in three orthogonal directions. Calculations by computer generalizing those due to Miksis *et al.* (1981) seem to offer the best prospect for illuminating the present conjectures.

As regards the generation of spiralling motions when I_1 is made large enough, the following explanation is plausible. Two eventualities have to be appreciated. First, a non-zero impulsive couple ($|L_1| \neq 0$) may be acquired simultaneously from the initial, typically non-axisymmetric process of formation imparting $I_1 > 0$, as when a small bubble is squeezed out of a nozzle and its velocity of rise increases until soon the buoyancy force is balanced by drag. This eventuality seems to have prevailed in the experiments by Hartunian & Sears (1957).

Second, a zigzag motion may arise initially which, if I_1 becomes too large, will amplify to beyond reach of stable limit cycles. Then a disruptive process ensues, which ends with some fraction of linear impulse and fraction of energy being shed in a concentrated motion that is left behind and carries an impulsive couple $-L_1$ relative to the original axis, while the bubble proceeds in a spiral path characterized by the remainder of I_1 and an impulsive couple $L_1 > 0$. Thus, in effect, the bubble is transmuted from an unstable state near the maximum of C_0 in figure 5 into a stable spiralling orbit represented somewhere to the right and a little lower in the figure, this transformation necessarily causing some of the original energy to be left behind. One possibility for such a process is that a smaller bubble is broken off, finally moving along or spirally about a straight line that does not intersect the axis of the main spiral. A more likely possibility is that a vortex ring is shed, finally moving along such a line.

6. Multiply connected bubbles

All the preceding theory depends on the assumption that Δ is a simply connected region, and the limitations so implied have been acknowledged at the end of §2 and the end of §3. Here a means of freeing the theory from these limitations is summarized. Note first, incidentally, that the Hamiltonian formulation explained in §2 admits immediate extension to the case of several separate bubbles, each of which is simply connected. The array of bubbles can be represented parametrically by taking (α, β) to range over respective disjoint regions of $\mathbb{R}^2$, and otherwise the formulation proceeds as before. In particular, the only modification of the conservation laws is that the bubbles contribute severally to integral properties such as E and $\boldsymbol{I}$.

Now, the theoretical and experimental study by Benjamin & Ellis (1966) demonstrated that a bubble at first simply connected will evolve into toroidal form when the impulse $\boldsymbol{I}$ of its motion, an invariant vector, has a magnitude sufficiently large in relation to the bubble's volume $\mathscr{V}$. (More specifically, for the state where W is maximum as discussed above, we have $I_3 = 3.57\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\mathscr{V}^{\frac{5}{6}}$; so values much larger than this one are in question.) The archetypal case is that of an empty bubble starting approximately spherical but then collapsing under a constant pressure $P_\infty > 0$ at infinity in the liquid, which case is covered by the present model with $P = -P_\infty$ and thus $E = K + \sigma|S| + P_\infty \mathscr{V}$. If $\boldsymbol{I} \neq 0$ initially, the bubble has to become toroidal as the only way of sustaining $\boldsymbol{I}$ as $\mathscr{V} \to 0$. The process whereby a jet of liquid penetrates the bubble in the direction of $\boldsymbol{I}$ is not wholly energy-conserving, however, and thus the description by Hamiltonian theory is interrupted. A revised Hamiltonian model may nevertheless apply reasonably well to the later motion, which may be supposed irrotational but with circulation around irreducible circuits in the liquid.

A suitable extension of the preceding theory can be derived from material in Lamb's book (1932, §§47–55, 132, 133, 139). The account there deals with rigid bodies moving in an infinite liquid, but relevant details are easily enough adaptable to the case of deformable interior boundaries S. It will suffice for illustration to consider a single toroidal bubble: thus $\Delta(t)$ bounded by $S(t)$ is such that the order of connectivity of $D(t) = \mathbb{R}^3 \backslash \Delta(t)$ is just two. A single cyclic constant κ must then be specified in determining the velocity potential ϕ. Following Lamb (1932, §132) one writes

$$\phi = \phi' + \kappa\chi, \tag{6.1}$$

where both ϕ' and χ are harmonic functions in $D(t)$ vanishing together with their gradients as $r \to \infty$ and satisfying conditions as follows. First, ϕ' is single-valued in D. Second, χ is a cyclic function satisfying $\partial\chi/\partial n = 0$ on S and differing by unity on the two sides of a fictitious surface ('barrier') B such that $D \backslash B$ is simply connected. As before, one of the Hamiltonian variables is taken to be $\Phi = \phi_S$. At each t, with S specified by the other dependent variable (i.e. $\boldsymbol{X}(\alpha, \beta, t)$ according to §2, $R(\alpha, \beta, t)$ according to §3), Φ in fact determines both ϕ' and χ when χ is subject to the stated conditions with a chosen value of the invariant parameter κ. This strategy can be generalized in the standard way when the order of connectivity of D is any integer $m \geqslant 2$; then $m-1$ cyclic constants are disposable (Lamb 1932, §§49, 50).

By appeal to Kelvin's extension of Green's theorem (Lamb 1932, §53) it is found that, for fixed κ, the first variation of K has no contribution from B, so taking precisely the same form as derived in §2 or §3. Thus the Hamiltonian representation of the modified dynamical problem is the same as the original, and the same correspondence holds between symmetries and conservation laws. The only difference is the implicit but crucial one, namely that Φ continually depends on the decomposition (6.1) of ϕ.

Hence variational principles for steady motions in the present class can at once be recognized. In particular, there is a characterization of uniformly translating toroidal bubbles with circulation, which are in effect hollow vortex rings. For a given κ, delimiting ϕ to accord with (6.1) and the conditions on χ, one considers the axisymmetric toroidal form of S and the axisymmetric Φ that minimize E_0 for given

$\mathscr{V}$ and given axial impulse I_3 with the same sign as κ. The important new feature is that I_3 has a component proportional to κ (cf. Lamb 1932, §133, (6)) whereby arbitrary positive values of I_3/κ can be attained, however small the value of $\mathscr{V}$. Specifically, this component of I_3 is expressible as a surface integral over B, whose span perpendicular to the axis x_3 can be made arbitrarily wide by adjusting the toroidal surface S. The variational principle compares with a known one for vortex rings, which is posed in terms of the Stokes stream function and for which κ is identified with the cross-sectional integral of azimuthal vorticity divided by cylindrical radius (Benjamin 1976).

Motions in which an asymmetric toroidal bubble spins steadily as well as translating are also describable by a variational principle. The constraint that L_3 has a given non-zero value is added to those of the preceding principle.

7. Concluding remarks

A moderately full examination of Hamiltonian theory for bubbles has been presented, with emphasis on the relation between symmetries and conservation laws and on group-invariant solutions which represent steady motions. The more general, parametric theory developed in §2 is in several respects more satisfactory than the alternative in §3, notwithstanding the greater formal complication of the former. But both approaches expose points of special interest and both warrant further attention, as also does the connection between them.

The variational characterizations of steady rectilinear, spinning and spiralling motions appear to be new, and they offer considerable scope for further calculations. Particular interest attaches also to the likelihood of bifurcation near the maximum of the Weber number W for axisymmetric steady motions, and to the questions about stability touched upon in §5.2.

I am indebted to Professor D. W. Moore who, at a seminar in Oxford four years ago, aroused my interest in the topic of this paper. Some time after he kindly gave me a copy of some unpublished notes of his on the problem of spiralling bubbles. Many of the analytical results recorded in §5 had been found by him using different means. I am grateful also to Mr Richard Fearn for undertaking the numerical computations that provided figures 2, 4 and 5.

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